Ark. Mat., **34** (1996), 383–406

On regularization in Banach spaces

Thomas Strömberg

1. Introduction

In the present paper we propose a regularization procedure for functions defined on Banach spaces admitting equivalent locally uniformly rotund norms the dual norm of which are also locally uniformly rotund. We demonstrate that with any bounded below lower semi-continuous (l.s.c.) proper function f defined on such a Banach space X can be associated a family of C^1 functions approximating f from below and enjoying favorable properties from the viewpoint of minimization. Our method reduces in the case where X is a Hilbert space to the one that was introduced and investigated by J. M. Lasry and P. L. Lions in their joint paper [10]. Their approach has subsequently been further explored by other authors, notably in [3] and [5], but never before outside the Hilbert space setting.

The Lasry–Lions method is based upon Moreau–Yosida approximation. Given f, an extended-real-valued function defined on a Hilbert space X, the *Moreau–Yosida approximates* of f are the functions f_t, $t>0$, that carry each $x\in X$ to

$$f_t(x)=\inf_{y\in X}\left(f(y)+\frac{1}{2t}\|x-y\|^2\right). \tag{1}$$

In the case where f is convex, l.s.c., and proper, the envelope functions f_t possess Lipschitz continuous Fréchet differentials (in symbols: $f_t\in C^{1,1}$) and $f_t\to f$, at least pointwise, as $t\downarrow 0$. There is even convergence, in certain senses, of the differentials df_t to the subdifferential ∂f. Furthermore, the infimal value of f together with the set of minimizers, as well as the stationary points and values, are preserved. The convexity hypothesis can actually be weakened; it suffices to assume that $f+(2T)^{-1}\|\cdot\|^2$ is convex for some $T>0$ in which case $f_t\in C^{1,1}$ etc. when $t\in(0,T)$.

In order to extend at least some of these results to non-convex functions, Lasry and Lions introduced a two-parameter family of approximates by putting $f_{t,s}=-(-f_t)_s$, $0<s<t$. Let us, for simplicity, assume that f is bounded from below which guarantees that the functions $f_{t,s}$ are all real-valued. It was proved in [10],