Ark. Mat., **32** (1994), 99–120

Zak's theorem on superadditivity

Audun Holme and Joel Roberts

0. Introduction

This paper is concerned with an important theorem due to *Fyodor L. Zak*, which appeared in [Z1]: Let $X \subset \mathbf{P}^N$ be a (reduced and irreducible) subvariety. A *k-secant space* to X is a k-dimensional linear subspace of $\mathbf{P}^N$ which is spanned by $k+1$ points from X, the *k-secant variety* of X in $\mathbf{P}^N$ is the (closure of the) union of all the k-secant spaces of X. Zak denotes this space by $S^k(X)$, we shall also use a different terminology: Whenever X and Y are subvarieties of $\mathbf{P}^N$, we define their *join* XY in $\mathbf{P}^N$ as the closure of the union of all *lines* in $\mathbf{P}^N$ spanned by a point from X and a point from Y. This defines a commutative and associative operation on the set of subvarieties of $\mathbf{P}^N$, making it into a *commutative monoid*, see [Å] for details. We have $S^k(X) = X^{k+1}$.

Zak considers a *relative secant defect*, defined as

$$\delta_k(X) = \dim(X^k) + \dim(X) + 1 - \dim(X^{k+1}).$$

We shall always assume that the ground field is algebraically closed of characteristic zero. Under this assumption, Zak states the following

Theorem (Zak's Theorem on Superadditivity). *Let $X \subset \mathbf{P}^N$ be a nonsingular projective variety, such that $\delta_1(X) > 0$. Let p and q be integers such that $X^{p+q} \neq \mathbf{P}^N$. Then $\delta_{p+q}(X) \geq \delta_p(X) + \delta_q(X)$.*

The assumption that $\delta_1(X) > 0$ was not explicitly stated in the formulation of this theorem in Zak's paper referred to above, but it is quite clear from the introduction that only this case is considered. In fact, there are counterexamples to the asserted inequality (for $p=q=2$) if $\delta_1(X) = 0$, and also for singular varieties, see §2 below.

In the applications of this theorem in his paper [Z1], Zak uses the theorem above only in the case $q=1$.