

Respectful quasiconformal extension from dimension $n-1$ to n

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1. Introduction

Tukia and Väisälä proved in [20] that every quasiconformal self-homeomorphism f of \mathbf{R}^{n-1} with $n \geq 2$ can be extended to a quasiconformal self-homeomorphism F of $\mathbf{R}_+^n = \mathbf{R}^{n-1} \times [0, \infty)$ which, in addition, in the hyperbolic metric of $H^n = \mathbf{R}^{n-1} \times (0, \infty)$ is bi-Lipschitz and uniformly approximates arbitrarily closely a natural homeomorphic extension F_f of f . The main result of this paper, Theorem 3.1, is that if X_0 is the subset \mathbf{R}^p ($0 \leq p < n$) or \mathbf{R}_+^p ($1 \leq p < n$) of \mathbf{R}^{n-1} and if f respects X_0 , i.e., maps it onto itself, then F can be chosen to respect $X = X_0 \times [0, \infty)$. Following Siebenmann, we call this extension theorem respectful (to X). An easy consequence, Theorem 4.1, is that if we forgo the properties of F involving the hyperbolic metric, F can be prescribed on X . The respectful quasiconformal Schoenflies extension theorem allows us to use Theorem 3.1 in Section 5 to show that every locally quasiconformal (LQC) self-homeomorphism of \mathbf{R}^{n-1} respecting X_0 can be extended to an LQC self-homeomorphism of \mathbf{R}_+^n respecting X . Moreover, the extension can be prescribed on X . This result, which generalizes the non-respectful version of it proved by the author in [10], is needed in [13] when proving that a self-homeomorphism of an LQC manifold M which respects a closed locally LQC flat LQC submanifold Q of M can be respectfully approximated by LQC homeomorphisms, i.e., by ones respecting Q , also in the case where Q meets the boundary of M (in this approximation theorem dimension four must necessarily be excluded).

In the proof of our main result we follow the simplified version of the proof of [20] as indicated in [19; 7.1]. For their part, Tukia and Väisälä were inspired by Carleson's [4] quasiconformal extension method in the case $n \leq 4$. We first decompose H^n into similar pieces (parallelotopes) and give each piece an index in $\{1, \dots, 2^n\}$ such that pieces of the same index are disjoint. From the quasiconformality of f it follows that the restrictions of the homeomorphism F_f to slightly larger paral-