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Entropy numbers of tensor products of operators

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This paper estimates the entropy numbers of tensor products of operators, mainly in a global sense. Let $S \in L^{(e)}_{s,w}(E_1, F_1)$, $T \in L^{(e)}_{s,w}(E_2, F_2)$ be operators between the Banach spaces E_i, F_i $(i=1,2)$. Here $L^{(e)}_{s,w}$ denotes the quasi-normed operator ideal consisting of the bounded linear operators with an $l_{s,w}$-summable sequence of entropy numbers for $0<s<\infty$, $0<w\leq\infty$. The size of the sequence

$$(e_n(S \widehat{\otimes}_\alpha T)) \tag{0.1}$$

is studied in the scale of the Lorentz sequence spaces for tensor norms α. Upper and lower estimates for the parameters of this scale are obtained for the sequence (0.1) for operators between special Banach spaces. We determine in Section 1 the precise behaviour in the Lorentz scale under tensoring with respect to the Hilbert–Schmidt tensor product of Hilbert spaces. König [K1, Lemma 1] exhibited relative to this problem the first examples of the instability of the entropy number ideals under the projective tensor norm. In Section 3 some stability results are shown assuming cotype 2 conditions on the spaces involved. We also compute bounds in some cases for the instability in the Lorentz scale with the help of volume arguments. The corresponding "local" problem of evaluating the individual entropy numbers of $S \widehat{\otimes}_\alpha T$ in terms of the entropy numbers of S and T is subtler. We establish in Section 2 asymptotic bounds for the entropy numbers of tensored operators on the Schatten trace classes $c_p(l^2)$.

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