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A smooth pseudoconvex domain in $\mathbf{C}^2$ for which L^∞-estimates for $\bar\partial$ do not hold

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Let $\mathcal{D}$ be a smoothly bounded domain in $\mathbf{C}^n$. It is well known (see [HL] and [Ø]) that if $\mathcal{D}$ is strictly pseudoconvex then we can solve the $\bar\partial$-equation with estimates in L^p for any $1 \le p \le \infty$. It has also been known for some time that this is no longer true if $\mathcal{D}$ is merely pseudoconvex. Namely, Sibony [S2] found an example of such a domain in $\mathbf{C}^3$ where L^∞-estimates do not hold. The reader should also consult the paper [FS1] which contains a discussion of L^p-estimates in general and many counterexamples to this type of questions. However, all counterexamples known seem to treat the case $n \ge 3$ and L^p-estimates for $p > 2$.

In this paper we shall prove

Theorem 1. *There is a smoothly bounded Hartogs domain in* $\mathbf{C}^2$, *and a* $\bar\partial$-*closed* $(0,1)$-*form* g *in* $\mathcal{D}$, *which extends continuously to* $\overline{\mathcal{D}}$, *such that the equation* $\bar\partial u = g$ *has no bounded solution.*

Recall that a Hartogs domain is a domain of the form

$$\mathcal{D} = \{(z,w); |w| < e^{-\varphi(z)}\} \tag{1}$$

where φ is subharmonic. If e.g. φ is smooth in the disk and

$$\varphi = \frac{1}{2} \log \frac{1}{1-|z|^2}$$

near the boundary of the disk, then $\partial\mathcal{D}$ will be smooth.

There is a special reason why we are interested in the case $n=2$. The form g in Theorem 1 extends continuously to $\partial\mathcal{D}$. So, the same example shows that we don't have L^∞-estimates for $\bar\partial_b$ either. But in $\mathbf{C}^2$ there is a duality between $\bar\partial_b$ in L^∞ and in L^1. Therefore we get

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