Acta Math., 176 (1996), 283–294

On Leray's self-similar solutions of the Navier–Stokes equations

by

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1. Introduction

In the 1934 paper [Le] Leray raised the question of the existence of self-similar solutions of the Navier–Stokes equations

$$\left.\begin{aligned} u_t-\nu\Delta u+(u\cdot\nabla)u+\nabla p=0\\ \operatorname{div} u=0\end{aligned}\right\} \quad \text{in } \mathbf{R}^3\times(t_1,t_2), \tag{1.1}$$

where, as usual, $\nu>0$. These are the solutions of the form

$$u(x,t)=\frac{1}{\sqrt{2a(T-t)}}\,U\biggl(\frac{x}{\sqrt{2a(T-t)}}\biggr), \tag{1.2}$$

where $T\in\mathbf{R}$, $a>0$, and $U=(U_1,U_2,U_3)$ is defined in $\mathbf{R}^3$. (Hence u is defined in $\mathbf{R}^3\times(-\infty,T)$.) One also requires that certain natural energy norms of u are finite. If $U\not\equiv 0$, then u given by (1.2) develops a singularity at time $t=T$. The Navier–Stokes equations for u give the system

$$\left.\begin{aligned} -\nu\Delta U+aU+a(y\cdot\nabla)U+(U\cdot\nabla)U+\nabla P=0\\ \operatorname{div} U=0\end{aligned}\right\} \quad \text{in } \mathbf{R}^3 \tag{1.3}$$

for U (where we use y to denote a generic point in $\mathbf{R}^3$). The main result of this paper is that the only solution of (1.3) belonging to $L^3(\mathbf{R}^3)$ is $U\equiv 0$.

We make a few remarks regarding the integrability condition $U\in L^3(\mathbf{R}^3)$. If one requires that u defined by (1.2) has finite kinetic energy and satisfies the natural energy equality

$$\int_{\mathbf{R}^3}\tfrac12|u(x,t_1)|^2\,dx=\int_{\mathbf{R}^3}\tfrac12|u(x,t_2)|^2\,dx+\int_{t_1}^{t_2}\int_{\mathbf{R}^3}\nu|\nabla u(x,t)|^2\,dx\,dt \tag{1.4}$$