SPECIAL SOLUTIONS OF CERTAIN DIFFERENCE EQUATIONS.

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Let $f(x)$ be a solution of the difference equation

$$f(x+1) - f(x) = g(x) \tag{1}$$

for $x > 0$. $f(x)$ may be uniquely determined by prescribing its values arbitrarily for $0 < x \leqq 1$. For certain functions $g(x)$ however the solution f may also be characterized by simple properties, instead of prescribed values in an interval. A solution may e.g. be uniquely determined by its asymptotic behaviour for large x; this leads to the »Hauptlösung» of the difference equation, as defined by N. E. Nörlund in his »Vorlesungen über Differenzenrechnung». In special instances solutions have also been characterized by local properties. Thus it has been proved by H. Bohr, that for $g(x) = \log x$, all strongly convex solutions of (1) are of the form $f(x) = \text{const.} + \log \Gamma(x)$.[1] Here a function is called »strongly convex», if

$$f(\lambda x + (1-\lambda) y) \leqq \lambda f(x) + (1-\lambda) f(y) \quad \text{for} \quad 0 \leqq \lambda \leqq 1, \tag{2}$$

whereas »convexity» alone only implies, that

$$f\left(\frac{x+y}{2}\right) \leqq \frac{f(x)+f(y)}{2}.$$ [2]

An analogous result has been derived recently by A. E. Mayer:[3] The only convex solution of the functional equation $1/f(x+1) = x f(x)$ is given by

$$f(x) = \frac{1}{\sqrt{2}} \frac{\Gamma\left(\frac{x}{2}\right)}{\Gamma\left(\frac{x+1}{2}\right)}.$$

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[1] Cf. e. g. E. Artin: Einführung in die Theorie der Gammafunktion, or Courant: Differential and Integral Calculus, vol. II p. 325.

[2] Convexity + boundedness in some finite interval is equivalent to strong convexity. Cf. Hardy, Littlewood, Polya: Inequalities, p. 91.

[3] Konvexe Lösung der Funktionalgleichung $1/f(x+1) = x f(x)$, Acta mathematica 70, p. 59.