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A Minkowski problem for electrostatic capacity

by

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Dedicated to my father, Meyer Jerison

Introduction

The Minkowski problem is to find a convex polyhedron from data consisting of normals to the faces and their surface areas. The corresponding problem for convex bodies with smooth boundaries is to find the convex body given the Gauss curvature of its boundary as a function of the unit normal. There is a natural notion of variation of a convex domain introduced by Minkowski, which is essentially variation of the boundary of the domain in the direction normal to the domain. Under this definition, the first variation of the volume of a convex body is the surface area measure on its boundary. The purpose of this paper is to develop a theory analogous to the one for the Minkowski problem in which volume is replaced by electrostatic capacity and surface area is replaced by the first variation of capacity. This theory was proposed in the paper [J].

Let $N\geqslant 3$ and let Ω be a bounded, convex, open subset of $\mathbf{R}^N$. Let Ω' be the complement of the closure of Ω. The equilibrium potential of Ω is the continuous function U defined in $\bar{\Omega}'$ satisfying

$$\Delta U=0 \text{ in } \Omega' \quad \text{and} \quad U=1 \text{ on } \partial\Omega'$$

and such that U tends to zero at infinity. Let $n=N-1$ and define the dimensional constant

$$a_N=\frac{1}{(N-2)\operatorname{vol}(S^n)},$$

where $\operatorname{vol}(S^n)$ is the volume of the unit sphere S^n in $\mathbf{R}^N$. Then $a_N|x|^{2-N}$ is the fundamental solution to Laplace's equation, i.e., $\Delta a_N|x|^{2-N}=-\delta_0$. It is well known that U

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