ON SMALL SUMSETS IN AN ABELIAN GROUP

BY

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1. Introduction

Let G be an abelian group, A, B and C subsets of G. By $A+B$ we denote the set of all the elements $g \in G$ having at least one representation as a sum $g=a+b$ of an element $a \in A$ and an element $b \in B$. For each $g \in G$, the number of such representations is denoted as $\nu_g(A, B)$. Further, $H(C)$ will denote the subgroup of G consisting of all the elements $g \in G$ for which $C+g=C$, thus, $C+H(C)=C$. If $H(C) \neq \{0\}$ then C is said to be *periodic*, otherwise, *aperiodic*. Finally, $[C]$ denotes the number of elements in C.

In this paper, we shall determine the structure of those pairs (A, B) of non-empty finite subsets of G for which

$$[A+B] < [A] + [B]. \tag{1}$$

In view of Theorem 3.1 due to Kneser [4], [5] it suffices to consider the case that $A+B$ is aperiodic and

$$[A+B] = [A] + [B] - 1, \tag{2}$$

cf. Theorem 3.4. If (2) holds, $2 \leqslant [A] < \infty$, $2 \leqslant [B] < \infty$, then (Theorem 2.1) either $A+B$ is in arithmetic progression or $A+B$ is the union of a non-empty periodic set C' and a subset C'' of some $H(C')$-coset. On the basis of such information on $A+B$, one can study the structure of the pair (A, B) itself, see section 4. The final result is Theorem 5.1; here besides (2) it is assumed that $\nu_c(A, B)=1$ has a solution c in case $A+B$ is periodic. Theorem 5.1 completely determines the (rather complicated) structure of the pairs (A, B) satisfying (1), cf. the discussion at the end of section 5.

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