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The order of the top Chern class of the Hodge bundle on the moduli space of abelian varieties

by

TORSTEN EKEDAHL and GERARD VAN DER GEER

Stockholm University
Stockholm, Sweden

University of Amsterdam
Amsterdam, The Netherlands

1. Introduction

Let $\mathcal{A}_g/\mathbf{Z}$ denote the moduli stack of principally polarized abelian varieties of dimension g. This is an irreducible algebraic stack of relative dimension $\frac{1}{2}g(g+1)$ with irreducible fibres over $\mathbf{Z}$. The stack $\mathcal{A}_g$ carries a locally free sheaf $\mathbf{E}$ of rank g, the Hodge bundle, defined as follows. If A/S is an abelian scheme over S with 0-section s we get a locally free sheaf $s^*\Omega^1_{A/S}$ of rank g on S, and this is compatible with pullbacks. If $\pi: A\to S$ denotes the structure map it satisfies the property $\Omega^1_{A/S}=\pi^*(\mathbf{E})$, and we will consider its Chern classes $\lambda_i(A/S):=c_i(\Omega^1_{A/S})$ (in the Chow ring of S). These then are the pullbacks of the corresponding classes in the universal case $\lambda_i:=c_i(\mathbf{E})$. The Hodge bundle can be extended to a locally free sheaf (again denoted by) $\mathbf{E}$ on every smooth toroidal compactification $\tilde{\mathcal{A}}_g$ of $\mathcal{A}_g$ of the type constructed in [9], see Chapter VI, §4 there. By a slight abuse of notation we will continue to use the notation λ_i for its Chern classes.

The classes λ_i are defined over $\mathbf{Z}$ and give for each fibre $\mathcal{A}_g\otimes k$ rise to classes, also denoted λ_i, in the Chow ring $\mathrm{CH}^*(\mathcal{A}_g\otimes k)$, and in $\mathrm{CH}^*(\tilde{\mathcal{A}}_g\otimes k)$. They generate subrings ($\mathbf{Q}$-subalgebras) of $\mathrm{CH}^*_{\mathbf{Q}}(\mathcal{A}_g\otimes k)$ and of $\mathrm{CH}^*_{\mathbf{Q}}(\tilde{\mathcal{A}}_g\otimes k)$ which are called the *tautological subrings*.

It was proved in [11] by an application of the Grothendieck–Riemann–Roch theorem that these classes in the Chow ring $\mathrm{CH}^*_{\mathbf{Q}}(\mathcal{A}_g)$ with rational coefficients satisfy the relation

$$(1+\lambda_1+\ldots+\lambda_g)(1-\lambda_1+\ldots+(-1)^g\lambda_g)=1. \tag{1.1}$$

Furthermore, it was proved that λ_g vanishes in the Chow group $\mathrm{CH}_{\mathbf{Q}}(\mathcal{A}_g)$ with rational coefficients. The class λ_g does not vanish on $\tilde{\mathcal{A}}_g$. This raises two questions. First, since λ_g is a torsion class on $\mathcal{A}_g$ we may ask what its order is. Second, since λ_g up to torsion