ADDITIVE VOLUME INVARIANTS OF RIEMANNIAN MANIFOLDS

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Introduction

Let (M, g) be an analytic Riemannian manifold and $m \in M$ a point. It is known (see e.g. [2]) that the volume of a small geodesic ball with center m and radius r is given by a power series expansion

$$V_m(r) = V_0(r)(1 + B_2 r^2 + B_4 r^4 + \ldots + B_{2k} r^{2k} + \ldots)$$

where $V_0(r)$ is the volume of the Euclidean ball of the same dimension and radius. Here the "volume invariants" B_2, B_4, ... are analytic functions of $m \in M$, or, more specifically, they are scalar curvature invariants of orders 2, 4, ... respectively.

A. Gray and L. Vanhecke [4] have calculated the first three invariants B_2, B_4, B_6 in terms of the curvature tensor R, the Ricci tensor ϱ, the scalar curvature τ and their covariant derivatives. In the same work the following was proved:

Let (M, g) be an analytic Riemannian manifold such that $V_m(r) = V_0(r)(1 + O(r^6))$ for all $m \in M$, i.e. such that $B_2 = B_4 = 0$ identically. Then (M, g) is flat in each of the following cases: (a) $\dim M \leqslant 3$, (b) M has non-positive or non-negative Ricci curvature, (c) M is conformally flat, (d) M is a product of surfaces, (e) M is locally a product of classical symmetric spaces and symmetric spaces of rank 1, (f) under some other special conditions which we do not write down explicitly.

On the other hand, the following examples have been given:

(i) A 4-dimensional Riemannian manifold such that $R \neq 0$ and $V_m(r) = V_0(r)(1 + O(r^6))$ for all $m \in M$.

(ii) A 5-dimensional homogeneous Riemanian manifold such that $R \neq 0$ and $V_m(r) = V_0(r)(1 + O(r^6))$.

(iii) A direct product of non-flat homogeneous Riemannian manifolds of total dimension $n = 734$ and such that $V_m(r) = V_0(r)(1 + O(r^8))$.