MINIMAL TRIANGULATIONS ON ORIENTABLE SURFACES

BY

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Introduction

Let S be a compact 2-manifold. A polyhedron on S is called a *triangulation* if each face of the polyhedron is a triangle with 3 distinct vertices and the intersection of any two distinct triangles is either empty, a single vertex or a single edge (including the two vertices). A triangulation on S is called *minimal* if the number of triangles is minimal. For instance the tetrahedron is a minimal triangulation of the sphere and the well known embedding of the complete graph with 7 vertices in the torus is a minimal triangulation of the torus.

Let $\delta(S)$ be the number of triangles in a minimal triangulation of S. In 1950 at a seminar at the University of Bonn, E. Peschl mentioned the problem of determining $\delta(S)$ for each surface S. The question may well be older than this. In 1955 G. Ringel [9] gave a complete solution if S is non-orientable.

In this paper we present a complete solution of the orientable part of the problem. We prove a formula for $\delta(S_p)$ for the orientable surface S_p of genus p.

The proof of the formula is a problem similar in nature and at least equivalent in complexity to the problem of determining the genus of the complete graph K_n. In both problems one must exhibit triangular embeddings of graphs which are complete or nearly complete (where some edges are missing). In the genus problem for K_n one has to add handles in order to gain the missing edges. In the problem of determining $\delta(S_p)$ the situation is reversed: one must find ways to subtract handles in order to remove edges, while preserving the triangulation.

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