THE EQUATION $1^p + 2^p + 3^p + \cdots + n^p = m^q$

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1. Introduction

E. Lucas [22] has proved that the diophantine equation

$$1^2 + 2^2 + \cdots + n^2 = m^2$$

has only the two solutions $n = m = 1$; $n = 24$, $m = 70$ (cf. [19]). In this paper we consider the more general equation

$$S_p(n) = 1^p + 2^p + \cdots + n^p = m^q \tag{1.1}$$

where p and q are given positive integers, and positive integral solutions are required. Some cases of this equation have been discussed before (see [7], Ch. 1, 23) and a very few simple ones solved, but no general solution has, to our knowledge, been attempted.

A few algebraic properties of $S_p(n)$, some of them new, are reviewed in Section 2. Section 3 deals with certain numerical properties of $S_p(n)$ required subsequently.

The study of equation (1.1) is divided into two parts. In Section 4 it is considered from a general point of view, and it is proved that, for any given choice of p, q, the number of solutions is finite, unless one of the following is the case: $q = 1$; $p = 3$, $q = 2$ (trivial cases); $p = 1$, $q = 2$; $p = 3$, $q = 4$; $p = 5$, $q = 2$ (Theorem 1). Also a result concerning the number of solutions is obtained.

In Sections 5 (cases with q odd) and 6 (q even), the complete determination of the solutions is obtained in several cases by means of theorems concerning algebraic diophantine equations of several kinds. The cases in which the number of solutions is infinite reduce to "Pellian" equations and can be solved completely. For the rest,