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On isotropy irreducible Riemannian manifolds

by

MCKENZIE WANG([1]) and WOLFGANG ZILLER([2])

McMaster University
Hamilton, Ontario, Canada

University of Pennsylvania
Philadelphia, PA, U.S.A.

Introduction

A connected Riemannian manifold (M, g) is said to be isotropy irreducible if for each point $p \in M$ the isotropy group H_p, i.e. all isometries of g fixing p, acts irreducibly on $T_p M$ via its isotropy representation. This class of manifolds is of great interest since they have a number of geometric properties which follow immediately from the definition. By Schur's lemma the metric g is unique up to scaling among all metrics with the same isometry group. By the same argument, the Ricci tensor of g must be proportional to g, i.e. g is an Einstein metric. Furthermore, according to a theorem of Takahashi [Ta], every eigenspace of the Laplace operator of (M, g) with eigenvalue $\lambda \neq 0$ and of dimension $k+1$ gives rise to an isometric minimal immersion into $S^k(r)$ with $r^2 = \dim M/\lambda$, by using the eigenfunctions as coordinates (see Li [L] and § 6 of this paper for further properties of these minimal immersions). By a theorem of D. Bleecker [Bl], these metrics can also be characterised as being the only metrics which are critical points for every natural functional on the space of metrics of volume 1 on a given manifold.

From the definition it follows easily that the isometry group of g must act transitively on M. Hence (M, g) is also a Riemannian homogeneous space. Conversely, we can define a connected effective homogeneous space G/H to be isotropy irreducible if H is compact and Ad_H acts irreducibly on $\mathfrak{g}/\mathfrak{h}$. Given an isotropy irreducible homogeneous space G/H, there exists a G-invariant metric g, unique up to scaling, such that (M, g) is isotropy irreducible in the first sense. But if we start with a Riemannian manifold (M, g) which is isotropy irreducible, it can give rise to several

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