CRITICAL POINTS AND GRADIENT FIELDS OF SCALARS IN HILBERT SPACE.

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1. Introduction.

The paper is concerned with certain aspects of a theory of critical points of a scalar (i.e., a real valued function) $i(\mathfrak{x})$ defined in a Hilbert space H, especially with the relation of the critical points to the gradient field of $i(\mathfrak{x})$. Moreover, applications to the theory of non-linear integral equations are made.

Let V be a bounded open convex set of H, and S its boundary. We suppose that $i(\mathfrak{x})$ is defined in an open region V' containing $V + S$ in its interior. If $i(\mathfrak{x})$ is written in the form

$$i(\mathfrak{x}) = \|\mathfrak{x}\|^2/2 + I(\mathfrak{x}) \tag{1.1}$$

where $\|\mathfrak{x}\|$ denotes the norm in the space H we will always assume that $G(\mathfrak{x}) = \text{grad } I(\mathfrak{x})$ (Definition 2.2) exists and is completely continuous. Moreover, if a critical point is defined as a point $\mathfrak{x}$ for which

$$\text{grad } i(\mathfrak{x}) = \mathfrak{g}(\mathfrak{x}) = \mathfrak{x} + G(\mathfrak{x}) = \mathfrak{o} \tag{1.2}$$

it will be assumed that such a point is not degenerate (definition 3.2) and does not lie on the boundary S of V.

Under these assumptions it can be proved (theorem 3.1) that there are at most a finite number of critical points in V, say $\mathfrak{a}_1, \mathfrak{a}_2, \ldots \mathfrak{a}_s$. For each critical point $\mathfrak{a}_\sigma$ there will be defined a non-negative integer r_σ, the type of the critical point $\mathfrak{a}_\sigma$ (definition 4.1). It will be proved that the "quadratic form" giving the second differential at $\mathfrak{a}_\sigma$ can, by the use of a proper base of H, be written as a sum of squares multiplied by ± 1, the number of those multiplied