ANALYTIC FUNCTIONS WITH FINITE DIRICHLET INTEGRALS ON RIEMANN SURFACES

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Since around 1950 the general classification theory of Riemann surfaces has been studied. Although many fruitful results have been obtained, there are still unsolved fundamental problems in the theory concerning the spaces of analytic functions with finite Dirichlet integrals.

In this paper we shall be concerned with the following problems I and II (cf. [5, pp. 50–51]).

Problem I. Let $AD(R)$ be the complex linear space of analytic functions on a Riemann surface R with finite Dirichlet integrals. Does there exist a Riemann surface R satisfying $1 < \dim_{\mathbf{C}} AD(R) < \infty$?

Problem II. Let O_{AD} (resp. O_{ABD}) be the class of Riemann surfaces on which there are no nonconstant AD functions (resp. bounded AD functions). Does the strict inclusion relation $O_{AD} \subsetneqq O_{ABD}$ hold?

Let $HD(R)$ be the real linear space of harmonic functions on R with finite Dirichlet integrals. Then, it is known that for every natural number n there is a Riemann surface R satisfying $\dim_{\mathbf{R}} HD(R) = n$ (cf. [5, p. 197]). In contrast to this result, we show that $R \notin O_{AD}$ if and only if $\dim_{\mathbf{C}} AD(R) = \infty$.

Problem II has been open since the beginning of the study of the classification theory of Riemann surfaces. We show that the equality $O_{AD} = O_{ABD}$ holds. Moreover, we prove that the space $ABD(R)$, the space of bounded AD functions on a Riemann surface R, is dense in $AD(R)$ in the sense that for every $f \in AD(R)$ there is a sequence $\{f_n\} \subset ABD(R)$ such that $f_n(\zeta) = f(\zeta)$ for a fixed point $\zeta \in R$ and $\int_R |f_n' - f'|^2 dx dy \to 0$ $(n \to \infty)$.

This paper consists of three sections. The purpose of § 1 is to prove Proposition 1.9 concerning modifications of positive measures. Its proof is relatively long. This proposition is