ON FUNCTIONS ORTHOGONAL TO INVARIANT SUBSPACES

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Let H^2 denote the usual Hardy class of functions holomorphic in the unit disk, U. Let M denote a closed, invariant subspace of H^2. The theory of such subspaces is well-known and may be found, for example, in the first three chapters of Hoffman's book [6]; every such M has the form $M = \varphi H^2$, where $\varphi \in H^2$ is an *inner* function, $\varphi = Bs\Delta$ with

$$B(z) = \prod_{\nu=1}^{\infty} \left(-\frac{\bar{a}_\nu}{|a_\nu|}\right) \frac{z - a_\nu}{1 - \bar{a}_\nu z}, \quad s(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\sigma(\theta)\right\}$$

$$\Delta(z) = \exp\left\{-\sum_{\nu=1}^{\infty} r_\nu \frac{e^{i\theta_\nu} + z}{e^{i\theta_\nu} - z}\right\}$$

where $\{a_\nu\}$ is a Blaschke sequence $(\Sigma(1 - |a_\nu|) < \infty)$ $(\bar{a}_\nu/|a_\nu| \equiv 1$ is understood whenever $a_\nu = 0)$, σ is a finite, positive, continuous, singular measure, and $r_\nu \geq 0$, $\Sigma r_\nu < \infty$.

In this paper we study the subspace $M^\perp = H^2 \ominus M$. Our results may be summarized as follows: we obtain a unitary operator V which maps the sum of three L^2 spaces onto $M^\perp$. The first, corresponding to the factor B of φ, is the space $L^2(d\sigma_B)$, where σ_B is the measure on the positive integers that assigns a mass $1 - |a_k|$ to the integer k. The second L^2 space is $L^2(d\sigma)$, and the third is the sum of the L^2 spaces of Lebesgue measure on the real intervals of length r_j.

In the special case $\varphi = B$, the functions $h_n(z) = (1 - |a_n|^2)^{\frac{1}{2}} B_n(z)/(1 - \bar{a}_n z)$ (B_n the Blaschke product with zeros $a_1, \ldots, a_{n-1}$) form an orthonormal basis of $M^\perp$; cf. [10, p. 305], [1]. From this fact it follows easily that the map

$$V(\{c_n\})(z) = \sum_{n=1}^{\infty} c_n (1 + |a_n|)^{\frac{1}{2}} B_n(z) (1 - \bar{a}_n z)^{-1} (1 - |a_n|) \tag{0.1}$$

carries $L^2(d\sigma_B)$ isometrically onto $M^\perp$, and this represents one instance of our theorem.

(1) The first author's contribution was partially supported by N.S.F. Grant GP-6764.

(2) The second author's contribution was partially supported by N.S.F. Grant GP-9658.