ON THE GROWTH OF SLOWLY INCREASING UNBOUNDED HARMONIC FUNCTIONS

BY

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1. For a non-constant real-valued harmonic function $u(z)$ defined in a plane domain Ω let $l(c)$ denote the level set $\{z \in \Omega \mid u(z)=c\}$ and let $\Theta(c)=\int_{l(c)} |*du|$ where this is to have the value zero if $l(c)$ is void. Then if a is a value taken by u the behaviour of the integral $\int_a^b (\Theta(c))^{-1} dc$ as b tends to infinity provides important information about the rate of growth of u as we approach the boundary of Ω. In particular if u is bounded the integral will take the value infinity for a finite value of b. On the other hand, if u is unbounded above the more rapidly the function tends to infinity as we approach the boundary the more slowly the integral increases. Our attention is primarily fastened on those functions for which the integral is finite for all finite b but tends to infinity with b. In a sense they are the most slowly increasing unbounded harmonic functions.

Our principal method is the method of the extremal metric making use of the essential identity of the integral indicated and the module of the curve family made up of the $l(c)$ for $a<c<b$. We should point out the relationship to questions studied first by Hayman [3] and later by Eke [1, 2]. However, their use of the length–area method restricted their study to regular functions. As is usually the case the method of the extremal metric gives new and deeper insights while providing simpler proofs of more general results.

2. In order to avoid possible confusion, we shall state definitions of some terminologies used in the sequel.

By an *arc* we mean a one-to-one continuous mapping φ of one of the intervals $[0, 1]$, $[0, 1)$, $(0, 1]$, $(0, 1)$ into the Riemann sphere. We shall say that it *has an initial point* (or *terminal point*) if the "tail" $T_0=\bigcap_{\varepsilon>0} \mathrm{Cl}\,\{\varphi(t) \mid 0<t<\varepsilon\}$ (or $T_1=\bigcap_{\varepsilon>0} \mathrm{Cl}\,\{\varphi(t) \mid 1-\varepsilon<t<1\}$, resp.) consists of a single point; here the closure Cl is taken on the Riemann sphere.

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