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Sections of smooth convex bodies via majorizing measures

by

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Dedicated to Xavier Fernique on his 60th birthday

1. Introduction

A central line of research in convexity theory and local theory of Banach spaces is the problem, given a balanced convex set, to find sections of large dimension that are well behaved. The basic theorem in this direction is Dvoretzky's theorem that asserts that an n-dimensional balanced convex set C has sections of dimension at least $\log n$ that are nearly ellipsoids. This is optimal in general. When more regularity is assumed (in the form of cotype hypothesis on the jauge of C) much larger nearly Euclidean sections can be found, as was demonstrated in the landmark paper [FLM]. In a somewhat different direction but in the same spirit is Milman's theorem [M] asserting the existence of subspaces of quotients of finite-dimensional Banach spaces that are nearly Euclidean and of dimension proportional to the dimension of the space. The nearly Euclidean sections constructed in [FLM] are obtained by a random construction, that provides no information on the "direction" of the section. There are however situations where this information is essential. A typical case arises from harmonic analysis, when one considers a finite family of characters $(\gamma_i)_{i\in I}$ on (say) a compact group, and the space E they generate. In that case, not all the subspaces of E are equally interesting; those that are generated by a subset of the characters $(\gamma_i)_{i\in I}$ are translation invariant and of special interest. The starting point of this research is a theorem of Bourgain that asserts that one can find a subset J of I, with $\operatorname{card} J = (\operatorname{card} I)^{2/p}$, such that on the space generated by the characters $(\gamma_i)_{i\in J}$, the L_p and L_2 norms are equivalent. (The basic measure is of course the normalised Haar measure.) Roughly speaking, what Bourgain proved is the following.