K-theory for certain group C^*-algebras

by

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Introduction

The use of K-theoretic techniques in C^*-algebras has led to the solution of several outstanding problems, among them the conjecture of R. V. Kadison that $C_r^*(F_n)$, the reduced C^*-algebra of the free group on n generators ($n \geq 2$), has no nontrivial projections. This was resolved affirmatively by Pimsner and Voiculescu [6] as a corollary to a remarkable theorem which describes the K-groups for any reduced crossed product of a C^*-algebra by an action of a free group.

This paper originated in the author's attempt to understand the work of Pimsner and Voiculescu, and in particular to see whether their methods could be used to give a simpler proof that $K_0(C_r^*(F_n)) = \mathbf{Z}$. By slightly adapting their approach, we are able to give a description of $K_*(C_r^*(\Gamma))$ for any group Γ which is a free product of countable amenable groups (Corollary 5.5 below). When specialized to the case $\Gamma = F_n$, our results naturally agree with those of Pimsner and Voiculescu. Our proof of their result is not actually much simpler than theirs, given the technical simplifications that accrue from not considering crossed products, but we feel that the structure of the proof becomes clearer when displayed in a more general context.

The K-theory of the full C^*-algebras of some free product groups has been investigated by Cuntz [2] and Rosenberg [7]. Comparison of their results with ours shows that $K_*(C_r^*(\Gamma))$ is the same as $K_*(C^*(\Gamma))$ in all known cases.

A vital element in the work of Pimsner and Voiculescu is the construction of an extension, which they call the Toeplitz extension, of $C_r^*(F_n)$ by the algebra K of compact operators. The Toeplitz extension is intimately tied to the group of integers, and in order to be able to deal with free products of groups other than $\mathbf{Z}$ we have replaced it by another extension of $C_r^*(\Gamma)$ by K which can be constructed for any free product group Γ and which turns out to be somewhat easier to handle than the Toeplitz extension. We describe this extension and some of its properties in section three.

In section two we investigate what seems to us the crucial property of the integers