A NON-HARMONIC FOURIER SERIES.[1]

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Introduction.

A non-harmonic Fourier series in an expression of the type

$$\sum_n c_n e^{i\lambda_n \eta}, \quad -\pi \leq \eta \leq \pi, \tag{1}$$

in which the numbers λ_n $(n=0, \pm 1, \pm 2, \ldots)$ are not all integers. Paley and Wiener [6] began a systematic study of such series; and Levinson [5] continued their work. The central problem is to discover necessary and sufficient conditions upon the numbers $\{\lambda_n\}$ such that to each real function $f(\eta)$ of a given class there corresponds an expression of the type (1) summable to f for all or almost all η in $-\pi \leq \eta \leq \pi$. So far as I am aware, the best answer to this problem is due to Levinson ([5] Theorems XVIII and XIX), and is to this effect: if the λ_n are real, and if there exists a real constant D such that

$$|\lambda_n - n| \leq D < (p-1)/2p, \quad 1 < p \leq 2, \tag{2}$$

then to every $f(\eta)$ belonging to the Lebesgue class $L^p(-\pi, \pi)$ there corresponds a series (1) which is summable to $f(\eta)$ in the same sense as an ordinary Fourier series $\sum_n c'_n e^{in\eta}$; and that these conclusions are false for the set

$$\lambda_{-n} = -n + (p-1)/2p, \quad \lambda_0 = 0, \quad \lambda_n = n - (p-1)/2p, \quad n = 1, 2, \ldots \tag{3}$$

On account of this last clause, Levinson refers to (2) as a "best possible" result. This phrase is perhaps unfortunate; since, as we shall show, it is not true that every set $\{\lambda_n\}$ which violates (2) does not admit representations of type (1) for every function of $L^p(-\pi, \pi)$. Secondly Levinson's theorem does not cater for the class $L(-\pi, \pi)$, which is the appropriate class for ordinary Fourier analysis and which is wider than and includes the classes $L^p(-\pi, \pi)$ for $p > 1$. Thirdly Levinson's theorem does not admit complex numbers λ_n.

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