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A lattice version of the KP equation

by

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1. Introduction

Let N and M be relatively prime integers. Let $V_{N,M}$ be the set of all real valued functions ψ on $\mathbf{Z}\times\mathbf{Z}$ satisfying $\psi(n+N, m)=\psi(n, m+M)=\psi(n, m)$. $V_{N,M}$ is a vector space of dimension NM over $\mathbf{R}$. Let A and B be functions from an interval $I=(a, b)$ to $V_{N,M}$. $A(n, m, t)$ will denote the value of $A(t)$ at the point $(n, m)\in\mathbf{Z}\times\mathbf{Z}$. In § 3, we will define two explicit real polynomial maps $f_{N,M}$ and $g_{N,M}$ on $V_{N,M}\times V_{N,M}\times\mathbf{R}^3$ to $V_{N,M}$. We will investigate solutions $A(t)$ and $B(t)$ to the following differential-difference equation:

$$\frac{dA(t)}{dt}=f_{N,M}(A(t), B(t), \alpha, \beta, \gamma) \tag{1.1}$$

$$\frac{dB(t)}{dt}=g_{N,M}(A(t), B(t), \alpha, \beta, \gamma) \tag{1.2}$$

for fixed α, β and γ. More intrisically, one may think of $f_{N,M}$ and $g_{N,M}$ as defining a vector field on $V_{N,M}\times V_{N,M}$ depending on parameters α, β and γ. Thus for any given t, $f_{N,M}(A(t), B(t), \alpha, \beta, \gamma)$ is a function on $\mathbf{Z}\times\mathbf{Z}$, and this function evaluated at (n, m) is a polynomial in α, β and γ and the numbers $A(i, j, t)$ and $B(i, j, t)$ which will turn out to be of degree 4, and $g_{N,M}(A(t), B(t), \alpha, \beta, \gamma)$ will turn out to be of degree 5. Actually these polynomials enjoy certain homogeneity properties explained in § 3.

These equations are derived from a certain algebro-geometric construction, which is in some sense a variant of a construction of Mumford and van Moerbeke (as will be explained in § 3). This construction starts with certain algebraic curves X with a distinguished point P (with certain additional structure). Using X and this structure, we

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