Normal forms for real surfaces in $\mathbf{C}^2$ near complex tangents and hyperbolic surface transformations

by

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0. Introduction

It is well known that the complex analytical properties of a real submanifold M in the complex space $\mathbf{C}^n$ are most accessible through consideration of the complex tangents to M. The properties we have in mind are related to the behavior of holomorphic functions on or near M and to the behavior of M under biholomorphic transformation. The case in which M is a real hypersurface is most familiar, while much less is known for higher codimension. In this paper we consider the critical case of a real n-dimensional manifold M in $\mathbf{C}^n$, which we also assume to be real analytic. At a generic point M is locally equivalent to the standard $\mathbf{R}^n$ in $\mathbf{C}^n$. However, near a complex tangent M may aquire a non-trivial local hull of holomorphy and other biholomorphic invariants.

We begin with the simplest non-trivial case, which is a surface $M^2 \subset \mathbf{C}^2$ with an isolated, suitably non-degenerate complex tangent. Here one already encounters a rich structure and non-trivial problems. In coordinates $z_j = x_j + iy_j$, $j = 1, 2$, M may be written locally as

$$R(z, \bar{z}) = -z_2 + q(z_1, \bar{z}_1) + \dots = 0,$$

$$q = \gamma z_1^2 + z_1 \bar{z}_1 + \gamma \bar{z}_1^2, \quad 0 \leq \gamma < \infty.$$

The z_1-axis is tangent to M at the origin. M, or more precisely, this complex tangent is said to be elliptic if $0 \leq \gamma < 1/2$, hyperbolic if $1/2 < \gamma$, or parabolic if $\gamma = 1/2$. We shall prove the following theorem.

(1) Alfred P. Sloan Fellow. Partially supported by NSF, Grant No. MCS 8100793.