Acta Math., 191 (2003), 259–297

Torus actions on manifolds of positive sectional curvature

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1. Introduction

We present several new results on isometric torus actions on positively curved manifolds. The symmetry rank was introduced by Grove and Searle as one possible way to measure the amount of symmetry of a Riemannian manifold (M, g). It is defined as the rank of the isometry group,

$$\operatorname{symrank}((M,g)) = \operatorname{rank}(\operatorname{Iso}(M,g)),$$

or equivalently as the largest number d such that a d-dimensional torus acts effectively and isometrically on M.

Grove and Searle [13] showed that $\operatorname{symrank}((M,g)) \leqslant \left[\frac{1}{2}(n+1)\right]$ provided that M is a compact manifold of positive sectional curvature. They also studied the case of equality and showed that this can only occur if the underlying manifold is diffeomorphic to $\mathbf{CP}^n$, $\mathbf{S}^n$, or to a lens space.

Our main new tool is the following basic result.

THEOREM 1. *Let M^n be a compact Riemannian manifold with positive sectional curvature. Suppose that $N^{n-k} \subset M^n$ is a compact totally geodesic embedded submanifold of codimension k. Then the inclusion map $N^{n-k} \to M^n$ is $(n-2k+1)$-connected.*

Recall that a map $f: N \to M$ between two manifolds is called h-connected, if the induced map $\pi_i(f): \pi_i(N) \to \pi_i(M)$ is an isomorphism for $i<h$ and an epimorphism for $i=h$. If f is an embedding, this is equivalent to saying that up to homotopy M can be obtained from $f(N)$ by attaching cells of dimension $\geqslant h+1$. It is easy to find various examples where the conclusion of Theorem 1 is optimal. For example the 24-dimensional

The author is an Alfred P. Sloan Research Fellow and was partly supported by an NSF grant and a DFG fellowship.