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Convexity estimates for mean curvature flow and singularities of mean convex surfaces

by

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1. Introduction

Let $F_0: \mathcal{M} \to \mathbf{R}^{n+1}$ be a smooth immersion of a closed n-dimensional hypersurface of nonnegative mean curvature in Euclidean space, $n \geqslant 2$. The evolution of $\mathcal{M}_0 = F_0(\mathcal{M})$ by mean curvature flow is the one-parameter family of smooth immersions $F: \mathcal{M} \times [0, T[\to \mathbf{R}^{n+1}$ satisfying

$$\frac{\partial F}{\partial t}(p,t) = -H(p,t)\nu(p,t), \quad p \in \mathcal{M},\ t \geqslant 0, \tag{1.1}$$

$$F(\cdot, 0) = F_0, \tag{1.2}$$

where $H(p,t)$ and $\nu(p,t)$ are the mean curvature and the outer normal respectively at the point $F(p,t)$ of the surface $\mathcal{M}_t = F(\cdot, t)(\mathcal{M})$. The signs are chosen such that $-H\nu = \vec{H}$ is the mean curvature vector and the mean curvature of a convex surface is positive.

For closed surfaces the solution of (1.1)–(1.2) exists on a finite maximal time interval $[0, T[$, $0 < T < \infty$, and the curvature of the surfaces becomes unbounded for $t \to T$. It is important to obtain a detailed description of the singular behaviour for $t \to T$, a future goal being the topologically controlled extension of the flow past singularities.

In the present paper we use the assumption of nonnegative mean curvature to derive new a priori estimates from below for all other elementary symmetric functions of the principal curvatures, strong enough to conclude that any rescaled limit of a singularity is (weakly) convex.

Let $\vec{\lambda} = (\lambda_1, ..., \lambda_n)$ be the principal curvatures of the evolving hypersurfaces $\mathcal{M}_t$, and let

$$S_k(\lambda) = \sum_{1 \leqslant i_1 < i_2 < ... < i_k \leqslant n} \lambda_{i_1} \lambda_{i_2} ... \lambda_{i_k}$$