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A characterization of product *BMO* by commutators

by

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1. Introduction

In this paper we establish a commutator estimate which allows one to concretely identify the *product BMO* space, $BMO(\mathbf{R}_+^2\times\mathbf{R}_+^2)$, of A. Chang and R. Fefferman, as an operator space on $L^2(\mathbf{R}^2)$. The one-parameter analogue of this result is a well-known theorem of Nehari [8]. The novelty of this paper is that we discuss a situation governed by a two-parameter family of dilations, and so the spaces H^1 and BMO have a more complicated structure.

Here $\mathbf{R}_+^2$ denotes the upper half-plane and $BMO(\mathbf{R}_+^2\times\mathbf{R}_+^2)$ is defined to be the dual of the real-variable Hardy space H^1 on the product domain $\mathbf{R}_+^2\times\mathbf{R}_+^2$. There are several equivalent ways to define this latter space, and the reader is referred to [5] for the various characterizations. We will be more interested in the biholomorphic analogue of H^1, which can be defined in terms of the boundary values of biholomorphic functions on $\mathbf{R}_+^2\times\mathbf{R}_+^2$ and will be denoted throughout by $H^1(\mathbf{R}_+^2\times\mathbf{R}_+^2)$, cf. [10].

In one variable, the space $L^2(\mathbf{R})$ decomposes as the direct sum $H^2(\mathbf{R})\oplus\overline{H^2(\mathbf{R})}$, where $H^2(\mathbf{R})$ is defined as the boundary values of functions in $H^2(\mathbf{R}_+^2)$ and $\overline{H^2(\mathbf{R})}$ denotes the space of complex conjugate of functions in $H^2(\mathbf{R})$. The space $L^2(\mathbf{R}^2)$, therefore, decomposes as the direct sum of the four spaces $H^2(\mathbf{R})\otimes H^2(\mathbf{R})$, $\overline{H^2(\mathbf{R})}\otimes H^2(\mathbf{R})$, $H^2(\mathbf{R})\otimes\overline{H^2(\mathbf{R})}$ and $\overline{H^2(\mathbf{R})}\otimes\overline{H^2(\mathbf{R})}$, where the tensor products are the Hilbert space tensor products. Let $P_{\pm,\pm}$ denote the orthogonal projection of $L^2(\mathbf{R}^2)$ onto the holomorphic/anti-holomorphic subspaces, in the first and second variables, respectively, and let H_j denote the one-dimensional Hilbert transform in the jth variable, $j=1,2$. In terms of the projections $P_{\pm,\pm}$,

$$H_1=P_{+,+}+P_{+,-}-P_{-,+}-P_{-,-}\quad\text{and}\quad H_2=P_{+,+}+P_{-,+}-P_{+,-}-P_{-,-}.$$

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