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On the topology of spaces of holomorphic maps

by

JENS GRAVESEN(1)

IMFUFA, Roskilde University Centre
Roskilde, Denmark

1. Introduction

Let X and Y be two complex manifolds and form the two spaces $\mathrm{Hol}(X, Y)$ and $\mathrm{Map}(X, Y)$ of respectively holomorphic and continuous maps $X \to Y$, equipped with the compact-open topology.

We will study the inclusion of $\mathrm{Hol}(X, Y)$ into $\mathrm{Map}(X, Y)$ in the case, where X is a Riemann surface and Y is a generalized flag manifold or a loop group.

Let $\mathrm{Hol}_n^*(X, Y)$ and $\mathrm{Map}_n^*(X, Y)$ denote the spaces of based maps of degree n. In [14] G. Segal shows that the inclusion of $\mathrm{Hol}_n^*(X, \mathbf{CP}^m)$ into $\mathrm{Map}_n^*(X, \mathbf{CP}^m)$ is a homology equivalence up to dimension $(n-2g)(2m-1)$, where g is the genus of X. Segal conjectured that a similar statement holds, if $\mathbf{CP}^m$ is replaced by a flag manifold or a Grassmannian, and this was confirmed by M. A. Guest, [7], and F. C. Kirwan, [9].

If G is a compact Lie group, the loop group ΩG has many properties similar to a Grassmannian, see [12]. So it is natural to try to extend Segal's result to the inclusion of $\mathrm{Hol}_n^*(X, \Omega G)$ into $\mathrm{Map}_n^*(X, \Omega G)$, and this is indeed the purpose of this work.

Let $\mathcal{V}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_{\mathbf{C}})$ be the space of based isomorphism classes of holomorphic $G_{\mathbf{C}}$-bundles over $X\times\mathbf{CP}^1$, trivial over the axis $X\vee\mathbf{CP}^1$ and with characteristic class n. In [1] M. F. Atiyah describes how there is an imbedding of $\mathrm{Hol}_n^*(X, \Omega G)$ into $\mathcal{V}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_{\mathbf{C}})$.

The main result (Theorem 7.8) is that

$$\lim_{n\to\infty} H_*(\mathcal{V}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_{\mathbf{C}})) = H_*(\mathrm{Map}_0^*(X, \Omega G)).$$

(1) Now at Mathematical Institute, Technical University of Denmark, Building 303, DK-2800 Lyngby, Denmark.