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# Bounded orthogonal systems and the $\Lambda(p)$-set problem

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## 0. Introduction

Let $G$ be a compact Abelian group and $\Gamma$ the dual group of $G$. For $p>2$, a subset $\Lambda$ of $\Gamma$ is called a $\Lambda(p)$-set, provided $L^p_\Lambda(G)=L^2_\Lambda(G)$. Here $L^p_\Lambda\equiv L^p_\Lambda(G)$ denotes the closure in $L^p(G)$ of the characters belonging to $\Lambda$ and considered as functions on $G$. The reader will find an introduction to the subject in W. Rudin's 1960 paper [Ru] and the book of Lòpez and Ross [L-R].

The main problem in this area is to construct $\Lambda(p)$-sets which are not $\Lambda(r)$ for some $r>p$. This has so far only been done for $p$ an even integer. In this case, the $L^p$-norm may be expressed in an algebraic way and the solution is of an arithmetic or combinatorial nature. In this paper, we consider the range $2<p<\infty$. Our approach is the point of view of general uniformly bounded orthogonal systems and no further properties of characters are exploited. The main result is the following fact.

THEOREM 1. *Let $\Phi=(\varphi_1,\ldots,\varphi_n)$ be a sequence of n mutually othogonal functions, uniformly bounded by* 1 *(i.e., $\|\varphi_i\|_\infty\leq 1$, $i=1,\ldots,n$). Let $2<p<\infty$. There is a subset $S$ of $\{1,\ldots,n\}$, $|S|>n^{2/p}$ satisfying*

$$\left\|\sum_{i\in S} a_i\varphi_i\right\|_p \leq C(p)\left(\sum_{i\in S}|a_i|^2\right)^{1/2} \tag{0.1}$$

*for all scalar sequences $(a_i)$. Here $C(p)$ is a constant only dependent on p. In fact,* (0.1) *holds for a generic set $S$ of size $[n^{2/p}]$.*

Observe that the size $n^{2/p}$ is optimal. Indeed, if one considers for instance a finite Cantor group $G=\{1,-1\}^k$ and let $\Phi=G^*$, the space $L^p_S(G)$ is a Hilbertian subspace of