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Quasiconformal maps of cylindrical domains

by

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1. Introduction

1.1. We shall consider domains $D\subset\mathbf{R}^3$ which are of the form $G\times\mathbf{R}^1$ where G is a domain in the plane $\mathbf{R}^2$. The main problem considered in this paper is: When is $G\times\mathbf{R}^1$ quasiconformally equivalent to the round ball B^3? It is well known that this is true if G is the disk B^2. Indeed, the sharp lower bound $q_0=K_O(B^2\times\mathbf{R}^1)$ for the outer dilatation $K_O(f)$ for quasiconformal maps $f\colon B^2\times\mathbf{R}^1\to B^3$ is explicitly known:

$$q_0=\frac{1}{2}\int_0^{\pi/2}(\sin t)^{-1/2}\,dt=1.31102\ldots;$$

see [GV, Theorem 8.1]. We shall show that there is a quasiconformal map $f\colon G\times\mathbf{R}^1\to B^3$ if and only if G satisfies the *internal chord-arc condition*, which is recalled in Section 4 of this paper. It implies that the boundary of G is rectifiable.

We also show that if G is bounded then $K_O(f)\geqslant q_0$, and the equality is possible only if G is a round disk. For unbounded domains the corresponding lower bound is trivially one, which is attained when G is a half plane.

It is of some interest to note that although the result deals solely with quasiconformality, its proof will involve two other classes of maps: the locally bilipschitz maps and the quasisymmetric maps, the latter notion considered in a suitable metric of the product space $\partial^*G\times\mathbf{R}^1$ where ∂^*G is the prime and end boundary of G.

The main result is proved in Section 5 and the dilatation estimate in Section 6. Before that we give preliminary results on John domains, quasisymmetric maps, prime ends and chord-arc conditions. The following auxiliary results may have independent interest: Theorem 2.9 gives a useful condition for a weakly quasisymmetric map to be quasisymmetric. Theorem 2.20 gives a sufficient condition for a quasiconformal map to