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On a problem in simultaneous Diophantine approximation: Littlewood's conjecture

by

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Dedicated to Bill Parry on his 65th birthday

1. Introduction

1.1. Background: elementary number theory

Before stating the problem we recall a few fundamental results from the theory of Diophantine approximation. Given a real number x we use the standard notation $\|x\|$ to denote the distance of x to the nearest integer, and throughout I will denote the unit interval $[0,1]$. The classical result of Dirichlet states:

DIRICHLET'S THEOREM (1842). *For any $\alpha\in I:=[0,1]$, there exist infinitely many $q\in\mathbf{N}$ such that*

$$\|q\alpha\|\leqslant q^{-1}.$$

A consequence of Hurwitz's theorem is that the right-hand side of the above inequality cannot be improved by an arbitrary positive constant ε. More precisely, for $\varepsilon<1/\sqrt{5}$ there exist real numbers $\alpha\in I$ for which the inequality $\|q\alpha\|\leqslant\varepsilon q^{-1}$ has at most a finite number of solutions. These α are the badly approximable numbers, and we will denote by **Bad** the set of all such numbers; that is,

$$\mathbf{Bad}:=\{\alpha\in I: \text{there exist } c(\alpha)>0 \text{ so that } \|q\alpha\|>c(\alpha)q^{-1} \text{ for all } q\in\mathbf{N}\}.$$

We now briefly describe the beautiful connection between **Bad** and the theory of continued fractions. Let $\alpha=[a_1,a_2,a_3,...]$ represent the regular continued fraction expansion of α, and as usual let $p_n/q_n:=[a_1,a_2,a_3,...,a_n]$ denote its nth convergent. It is

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