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Sharp Lieb–Thirring inequalities in high dimensions

by

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0. Introduction

Let us consider a Schrödinger operator in $L^2(\mathbf{R}^d)$,

$$-\Delta+V, \tag{0.1}$$

where V is a real-valued function. Lieb and Thirring [23] proved that if $\gamma>\max\left(0,1-\frac{1}{2}d\right)$, then there exist universal constants $L_{\gamma,d}$ satisfying(1)

$$\operatorname{tr}(-\Delta+V)_-^\gamma \leqslant L_{\gamma,d}\int_{\mathbf{R}^d} V_-^{\gamma+d/2}(x)\,dx. \tag{0.2}$$

In the critical case $d\geqslant3$ and $\gamma=0$, the bound (0.2) is known as the Cwikel–Lieb–Rozenblum (CLR) inequality, see [8], [20], [25] and also [7], [19]. For the remaining case $d=1$ and $\gamma=\frac{1}{2}$, the estimate (0.2) has been verified in [27], see also [14]. On the other hand, it is known that (0.2) fails for $\gamma=0$ if $d=2$, and for $0\leqslant\gamma<\frac{1}{2}$ if $d=1$.

If $V\in L^{\gamma+d/2}(\mathbf{R}^d)$, then the inequalities (0.2) are accompanied by the Weyl-type asymptotic formula

$$\begin{aligned}\lim_{\alpha\to+\infty}\frac{1}{\alpha^{\gamma+d/2}}\operatorname{tr}(-\Delta+\alpha V)_-^\gamma &= \lim_{\alpha\to+\infty}\frac{1}{\alpha^{\gamma+d/2}}\iint_{\mathbf{R}^d\times\mathbf{R}^d}(|\xi|^2+\alpha V)_-^\gamma\,\frac{dx\,d\xi}{(2\pi)^d}\\ &= L^{\mathrm{cl}}_{\gamma,d}\int_{\mathbf{R}^d}V_-^{\gamma+d/2}\,dx,\end{aligned} \tag{0.3}$$

(1) Here and below we use the notion $2x_-:=|x|-x$ for the negative part of variables, functions, Hermitian matrices or self-adjoint operators.