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A characterization of all elliptic algebro-geometric solutions of the AKNS hierarchy

by

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1. Introduction

Before describing our approach in some detail, we shall give a brief account of the history of the problem of characterizing elliptic algebro-geometric solutions of completely integrable systems. This theme dates back to a 1940 paper of Ince [51] who studied what is presently called the Lamé–Ince potential

$$q(x) = -n(n+1)\wp(x+\omega_3), \quad n \in \mathbf{N},\ x \in \mathbf{R}, \tag{1.1}$$

in connection with the second-order ordinary differential equation

$$y''(E,x)+q(x)y(E,x) = Ey(E,x), \quad E \in \mathbf{C}. \tag{1.2}$$

Here $\wp(x)=\wp(x;\omega_1,\omega_3)$ denotes the elliptic Weierstrass function with fundamental periods $2\omega_1$ and $2\omega_3$ ($\mathrm{Im}(\omega_3/\omega_1)\neq 0$). In the special case where ω_1 is real and ω_3 is purely imaginary, the potential $q(x)$ in (1.1) is real-valued and Ince's striking result [51], in modern spectral-theoretic terminology, yields that the spectrum of the unique self-adjoint operator associated with the differential expression $L_2=d^2/dx^2+q(x)$ in $L^2(\mathbf{R})$ exhibits finitely many bands (and gaps, respectively), that is,

$$\sigma(L_2) = (-\infty, E_{2n}] \cup \bigcup_{m=1}^{n} [E_{2m-1}, E_{2m-2}], \quad E_{2n} < E_{2n-1} < \dots < E_0. \tag{1.3}$$

What we call the Lamé–Ince potential has, in fact, a long history and many investigations of it precede Ince's work [51]. Without attempting to be complete we refer the

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