

# Harmonic analysis, cohomology, and the large-scale geometry of amenable groups

by

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To my father Abraham, and to our baby daughter Noga

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