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Quasiconformal 4-manifolds

by

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§1. Introduction

For any pseudo-group of homeomorphism of Euclidean space one can define the corresponding category of manifolds. The most familiar examples in Topology are the full pseudo-group of homeomorphisms, giving rise to the theory of topological manifolds, and the subgroup of smooth differomorphisms giving rise to the theory of C^∞ manifolds. In this paper, we discuss an intermediate category—*quasiconformal* homeomorphisms and manifolds.

Recall that a homeomorphism $\varphi:D\to\mathbf{R}^n$ from a domain D in $\mathbf{R}^n$ to its image $\varphi(D)$ is K quasiconformal if for all x in D

$$H_\varphi(x)=\limsup_{r\to 0}\frac{\max\{|\varphi(y)-\varphi(x)|\mid |y-x|=r\}}{\min\{|\varphi(y)-\varphi(x)|\mid |y-x|=r\}}\leqslant K.$$

φ is quasiconformal (QC) if it is K quasiconformal for some $K\geqslant 1$. Roughly, a quasiconformal map distorts the relative distances of nearby points by a bounded factor. Contrast this with the *Lipschitz* condition: a homeomorphism φ is bi-*Lipschitz* if for some $C\geqslant 1$ and all x,y in D:

$$C^{-1}|x-y|\leqslant|\varphi(x)-\varphi(y)|\leqslant C|x-y|.$$

Both these conditions define pseudo-groups of homeomorphism and hence *quasiconformal* and *Lipschitz n-manifolds*; Hausdorff spaces made from domains in $\mathbf{R}^n$ pieced together by, respectively, quasiconformal and Lipschitz homeomorphisms. We also have the obvius notions of *equivalence* in the two categories.