On the Stokes conjecture for the wave of extreme form

by

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1. Introduction

1.1. The Stokes conjecture

In this paper we settle a question of the regularity, at one exceptional point, of the free boundary in a problem governed by the Laplace equation and a non-linear boundary condition.

The physical problem concerns gravity waves of permanent form on the free surface of an ideal liquid (that is, of a liquid having constant density, no viscosity and no surface tension). We suppose throughout that the motion is two-dimensional, irrotational and in a vertical plane. Of the various cases to be introduced in section 2, we consider here only the simplest: that of periodic waves on liquid of infinite depth. If we take axes moving with the wave (axes fixed relative to a crest) as in Figure 1, the problem becomes one of steady motion; the fluid domain is

$$\Omega = \{ (x, y): -\infty < x < \infty,\ -\infty < y < Y(x) \},$$

where the free surface $\Gamma = \{(x, Y(x)) \colon x \in \mathbf{R}\}$ is unknown a priori, and Y is to have period λ. Moreover, we assume Γ to have a single crest (Y to have a single maximum) per wavelength, and Γ to be symmetrical about that crest. One seeks a stream function Ψ that (a) is harmonic ($\Delta\Psi = 0$) in Ω, (b) satisfies $\Psi(x+\lambda, y) = \Psi(x, y)$, (c) is such that the fluid velocity $(\Psi_y, -\Psi_x) \to (c, 0)$ as $y \to -\infty$, (d) satisfies the free-surface conditions

$$\Psi = 0 \quad \text{and} \quad \frac{1}{2}|\nabla\Psi|^2 + gy = \text{constant on } \Gamma. \tag{1.1}$$

Here the wavelength λ and gravitational acceleration g are given positive constants, and the wave velocity $(-c, 0)$, relative to the fluid at infinite depth, is to be found after