Acta Math., 179 (1997), 79–103

Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds

by

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1. Introduction

In a solid medium, heat flow is governed by two characteristics, conductivity and capacity, which may vary over the medium, sometimes in an irregular way. A general mathematical model is provided by a manifold M, in which the conductivity, or rather its inverse, the resistance, corresponds to a Riemannian metric, and the capacity corresponds to a Borel measure m. We shall be concerned with the heat flow on M associated to the Dirichlet form

$$\mathcal{E}(f)=\int_M |\nabla f|^2\,dm.$$

In particular, we shall consider the Dirichlet heat kernel $p_0(t,x,y)$ and the Neumann heat kernel $p(t,x,y)$. Physically, these express the rise in temperature at y after time t, due to unit heat input at x, when, respectively, the boundary is maintained at a fixed temperature or is perfectly insulated. Our main aim will be to relate, under minimal hypotheses, the small time asymptotics of these heat kernels to distance functions derived from the metric. We shall show that the basic asymptotic formula of Varadhan [V1] remains valid without smoothness assumptions on the metric or measure, and indeed without any sort of completeness or curvature bound on the underlying space.

We work throughout in the context of a Lipschitz Riemannian manifold M, of dimension n, on which is given a Borel measure m. See for example [DP1], [T], [Z]. We thus have a maximal atlas of charts, the transition functions between which are locally Lipschitz homeomorphisms in $\mathbf{R}^n$. Henceforth we shall write Lipschitz to mean locally Lipschitz. We systematically refer to a chart by its domain U, which is moreover then identified with its image, an open set in $\mathbf{R}^n$. We assume that in each chart U the measure m is absolutely continuous with respect to Lebesgue measure l in $\mathbf{R}^n$. In each chart U