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Degenerating the complex hyperbolic ideal triangle groups

by

RICHARD EVAN SCHWARTZ

University of Maryland
College Park, MD, U.S.A.

1. Introduction

A basic problem in geometry and representation theory is the *deformation problem*. Suppose that $\varrho_0: \Gamma \to G_1$ is a discrete embedding of a finitely generated group Γ into a Lie group G_1. Suppose also that $G_1 \subset G_2$, where G_2 is a larger Lie group. The deformation problem amounts to finding and studying discrete embeddings $\varrho_s: \Gamma \to G_2$ which extend ϱ_0.

Let $\mathbf{H}^2$ be the hyperbolic plane. The complex hyperbolic plane, $\mathbf{CH}^2$, is a complex 2-dimensional negatively curved symmetric space which contains $\mathbf{H}^2$ as a totally real, totally geodesic subspace, and is often considered to be its complexification. The theory of deforming $\mathrm{Isom}(\mathbf{H}^2)$-representations into $\mathrm{Isom}(\mathbf{CH}^2)$, while quite rich, is still in its infancy. (For a representative sample of such work, see [FZ], [GKL], [GuP], [KR], [To].) The state of affairs is such that one still needs to work out basic examples in detail to gain a foundation for more general considerations.

The *complex hyperbolic ideal triangle groups* are amongst the simplest concrete examples of complex hyperbolic deformations. A complex hyperbolic ideal triangle group is a representation of the form $\varrho_s: \Gamma \to \mathrm{Isom}(\mathbf{CH}^2)$. Here Γ is the free product $\mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$. The representation ϱ_s maps the standard generators to order-2 complex reflections, such that the product of any two unequal generators is parabolic. (See §2 for definitions.)

There is a real 1-parameter family $\{\varrho_s \mid s \in \mathbf{R}\}$ of nonconjugate complex hyperbolic ideal triangle groups. The representation ϱ_0 is the complexification of the familiar real ideal triangle group generated by reflections in the sides of an ideal geodesic triangle in the hyperbolic plane. The other representations are deformations.

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