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The colored Jones polynomials and the simplicial volume of a knot

by

HITOSHI MURAKAMI

Waseda University
Tokyo, Japan

and

The Mittag-Leffler Institute
Djursholm, Sweden

and

JUN MURAKAMI

Osaka University
Osaka, Japan

In [13], R. M. Kashaev defined a family of complex-valued link invariants indexed by integers $N \geqslant 2$ using the quantum dilogarithm. Later he calculated the asymptotic behavior of his invariant, and observed that for the three simplest hyperbolic knots it grows as $\exp(\mathrm{Vol}(K)N/2\pi)$ when N goes to infinity, where $\mathrm{Vol}(K)$ is the hyperbolic volume of the complement of a knot K [14]. This amazing result and his conjecture that the same also holds for any hyperbolic knot have been almost ignored by mathematicians since his definition of the invariant is too complicated (though it uses only elementary tools).

The aim of this paper is to reveal his mysterious definition and to show that his invariant is nothing but a specialization of the colored Jones polynomial. The colored Jones polynomial is defined for colored links (each component is decorated with an irreducible representation of the Lie algebra $\mathrm{sl}(2, \mathbf{C})$). The original Jones polynomial corresponds to the case that all the colors are identical to the 2-dimensional fundamental representation. We show that Kashaev's invariant with parameter N coincides with the colored Jones polynomial in a certain normalization with every color the N-dimensional representation, evaluated at the primitive Nth root of unity. (We have to normalize the colored Jones polynomial so that the value for the trivial knot is one, for otherwise it always vanishes.)

On the other hand, there are other colored polynomial invariants, such as the generalized multivariable Alexander polynomial defined by Y. Akutsu, T. Deguchi and T. Ohtsuki [1]. They used the same Lie algebra $\mathrm{sl}(2, \mathbf{C})$ but a different hierarchy of representations. Their invariants are parameterized by $c+1$ parameters: an integer N

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