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Primes represented by x^3+2y^3

by

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1. Introduction

It is conjectured that if $f(X)$ is any irreducible integer polynomial such that $f(1), f(2), \ldots$ tend to infinity and have no common factor greater than 1, then $f(n)$ takes infinitely many prime values. Unfortunately this has only been proved for linear polynomials, in which case the assertion is the famous theorem of Dirichlet. One may seek to formulate a weaker conjecture concerning irreducible binary forms $f(X,Y)$. Here the necessary condition is that the values of $f(m,n)$ for positive integers m,n are unbounded above and have no non-trivial common factor. Again one might hope that such a form attains infinitely many prime values. This is trivial for linear forms, as such a form takes all sufficiently large integer values. For quadratic forms it was proved by Dirichlet, although in certain special cases, such as $f(X,Y)=X^2+Y^2$, the result goes back to Fermat. Dirichlet's result was extended by Iwaniec [14] to quadratic polynomials in two variables. Our goal