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On equiresolution and a question of Zariski

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1. Introduction

Fix $x \in Y \subset V \subset W$ where x is a closed point, W is smooth over over the field $\mathbf{C}$ of complex numbers, V is a reduced hypersurface in W, and Y is an irreducible subvariety of V. Zariski proposes a notion of equisingularity intended to decide if the singularity at $x \in V$ is in some sense equivalent to that at $y \in V$, where y denotes the generic point of Y. In case the condition holds, we say that $x \in V$ and $y \in V$ are equisingular, or that V is equisingular along Y locally at x.

Zariski's notion relies and is characterized by two elementary properties, say (A) and (B).

(A) If $x \in V$ and $y \in V$ are equisingular, then $x \in V$ is regular if and only if $y \in V$ is regular.

Zariski formulates the second property in the algebroid context, namely at the completion of the local ring $\mathcal{O}_{W,x}$, say $R = \mathbf{C}[[x_1, ..., x_n]]$, a ring of formal power series over $\mathbf{C}$, and $n = \dim \mathcal{O}_{W,x}$. Assume for simplicity that Y is analytically irreducible at x (e.g. that Y is regular at x), and let y denote again the generic point of Y at R. By the Weierstrass preparation theorem one can define a formally smooth morphism

$$\pi: U_1 = \operatorname{Spec}(\mathbf{C}[[x_1, ..., x_n]]) \to U_2 = \operatorname{Spec}(\mathbf{C}[[x_1, ..., x_{n-1}]])$$

so that π induces a finite morphism $\pi: V \to U_2$. In such case let $D_\pi \in \mathbf{C}[[x_1, ..., x_{n-1}]]$ be the discriminant. Let $\Sigma_\pi = V(D_\pi) \subset U_2$ be the reduced hypersurface in U_2 defined by D_π (reduced discriminant). Note now that $\dim U_2 = \dim V = n-1$, and V is unramified over $U_2 - \Sigma_\pi$; so $\pi(y) \in \Sigma_\pi$ if V is singular at y.