Acta Math., 185 (2000), 81–99

Kelvin transform and multi-harmonic polynomials

by

JEAN-LOUIS CLERC

Université Henri Poincaré
Vandœuvre-lès-Nancy, France

En mémoire de ma mère

The classical *Kelvin transform* associates to a smooth function f on $\mathbf{R}^N\setminus\{0\}$ ($N\geqslant 2$) the function Kf (also defined on $\mathbf{R}^N\setminus\{0\}$) by the formula

$$Kf(\xi)=\|\xi\|^{2-N}f(\xi/\|\xi\|^2).$$

The main result is that if f is harmonic, then Kf is also harmonic. Although we shall not use this remark, this result reflects a covariance property of the Laplace operator under the action of the conformal group. The Kelvin transform is used to generate (all) harmonic polynomials on $\mathbf{R}^N$ by the following process (due to Maxwell, cf. [CH]): take p to be any polynomial on $\mathbf{R}^N$, form the usual constant-coefficient differential operator $\partial(p)$, apply it to the Green kernel $G(\xi)=\|\xi\|^{2-N}$ (to be replaced by $\log\|\xi\|$ in case $N=2$). The result $\partial(p)G$ is defined and harmonic on $\mathbf{R}^N\setminus\{0\}$, so that we may apply the Kelvin transform to get a harmonic function $K\partial(p)G$, which can be shown to extend to all of $\mathbf{R}^N$ as a harmonic polynomial. Moreover, all harmonic polynomials can be obtained by this process.

We generalize such a transformation in the context of analysis on matrix spaces. By this terminology is usually meant analysis that gives a special role to some subgroup of the linear or orthogonal group which can be interpreted as (say) a left action on a space of (rectangular) matrices. One classical case is the left action of $GL(n,\mathbf{R})$ on $\mathrm{Mat}_{n,m}(\mathbf{R})$, and related versions where $\mathbf{R}$ is replaced by $\mathbf{C}$ or the field of quaternions. Various notions of harmonic polynomials (under various names) were introduced in the literature (see the references at the end), related to invariance or covariance properties under the above-mentioned subgroup. Here we work in the context of representations of (Euclidean) Jordan algebras, where the appropriate theory of harmonic polynomials (called *Stiefel harmonics*) was developed in [C1]. It covers these classical cases, but also