THE PROBLEM OF UNITARY EQUIVALENCE.[1]

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The question of equivalence of matrices under the group G of unitary transformations has received attention from several writers [1, 2, 3, 4]. Fundamental in most investigations is the theorem of Schur [5] that any matrix A of complex numbers can be transformed by some unitary matrix into triangle form: (a_{ij}), $a_{ij}=0$ whenever $i>j$. A short proof of Schur's theorem appears in Murnaghan's book [6].

This theorem alone is not enough to settle the equivalence question; two matrices may be in triangle form, equivalent under G, and yet not equal. An example is given by the matrices $\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

If a matrix is in triangle form, the diagonal elements are the characteristic roots. Schur proves further that it is possible to find a unitary matrix U such that UAU^* is in triangle form and has its characteristic roots arranged in any order along the main diagonal. In order that two matrices be equivalent under G it is clearly necessary that they have the same characteristic roots; this condition is by no means sufficient.

This article investigates the question of equivalence under G:

$$A_1,\ B_1 \text{ given; } X \text{ to be found so that}$$

$$XA_1X^* = B_1, \quad XX^* = 1. \tag{1}$$

To solve problem (1), we follow a standard procedure: A_1 is transformed into a unique canonical form C_1. This canonical form will have the properties ordinarily ascribed to canonical forms. The definition of canonical form will be determinative; the canonical form will be unique; and the definition will be so arranged that two matrices equivalent under G have the same canonical form.

[1] The solution herewith presented was completed in 1948, but not published until now.