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Uniqueness of Kähler–Ricci solitons

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0. Introduction

In recent years, Ricci solitons have been studied extensively (cf. [Ha], [C2], [T2], etc.). One motivation is that they are very closely related to limiting behavior of solutions of PDE which arise in geometric analysis, such as R. Hamilton's Ricci flow equation and the complex Monge–Ampére equations associated to Kähler–Einstein metrics. Ricci solitons extend naturally Einstein metrics.

Let M be a compact Kähler manifold. A Kähler metric h with its Kähler form ω_h is called a Kähler–Ricci soliton with respect to a holomorphic vector field X if the equation

$$\mathrm{Ric}(\omega_h)-\omega_h=L_X\omega_h \tag{0.1}$$

is satisfied, where $\mathrm{Ric}(\omega_h)$ denotes the Ricci form of ω_h, and L_X is the Lie derivative operator along X (the definition here is slightly stronger than the ordinary one studied, for instance, in [C2]). Since ω_h is d-closed, we may write $L_X\omega_h=\partial\bar\partial\psi$ for some function ψ. It follows that the first Chern class $c_1(M)$ of M is positive and represented by ω_h.

If ω_h is a Kähler–Ricci soliton form with respect to a nontrivial X, then Futaki's invariant with respect to X is

$$F(X)=\int_M |X|^2\omega_h^n\neq 0.$$

By using a result in [F1], we see that there are no Kähler–Einstein metrics on M if M admits a Kähler–Ricci soliton with respect to a nontrivial X. Hence, the existence of Kähler–Ricci solitons is an obstruction to the existence of Kähler–Einstein metrics on compact Kähler manifolds with positive first Chern class. Examples of Kähler–Ricci solitons were found on certain compact Kähler manifolds by E. Calabi [Cal], N. Koiso [Koi] and H. D. Cao [C1], respectively.