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Local solvability for a class of differential complexes

by

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Introduction—The Treves conjecture

In 1983 F. Treves [T1] initiated the study of the local solvability for the class of differential complexes defined by a smooth, locally integrable structure of rank n in $\mathbf{R}^{n+1}$. If Z denotes a local first integral of the structure, Treves conjectured that the vanishing of the local cohomology in degree q of such a differential complex would be related to the vanishing of the singular homology of the sets $Z=\text{const.}$ in dimension $q-1$. It is the purpose of this article to complete the proof of this conjecture in its full generality.

(A) In order to motivate and state the problem more precisely we first recall the question of local solvability for a single vector field in two variables. Let thus

$$\mathrm{L}=a(x,t)\frac{\partial}{\partial t}+b(x,t)\frac{\partial}{\partial x}$$

be a complex vector field defined in a neighborhood X of the origin in $\mathbf{R}^2$ with no singularities. We say that L is *solvable* (at the origin) if the induced map $\mathrm{L}\colon\mathcal{C}_0^\infty\to\mathcal{C}_0^\infty$ is surjective, where we have written $\mathcal{C}_0^\infty$ to denote the space of germs of smooth functions at the origin.

The solvability of L is characterized by the so-called condition $(\mathcal{P})$ of Nirenberg–Treves (cf. [H, Chapter XXVI], [NT]). For the purpose of our presentation it is convenient to make an extra assumption and assume the *integrability* of L, in the sense that there exists $Z\in\mathcal{C}^\infty(X)$ such that $\mathrm{L}Z=0$ and $\mathrm{d}Z\neq 0$ at every point of X.

It takes an elementary argument to show that we can choose such a solution Z that can be written, in an appropriate coordinate system around the origin, as

$$Z(x,t)=x+i\varphi(x,t),$$

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