Extending holomorphic motions

by

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1. Introduction

The notion of an isotopy of one set within another set is one of the key concepts of topology. Here is one way this concept can be generalized to a holomorphic context:

Definition. If X is a subset of $\mathbf{C}$, a *holomorphic motion* of X in $\mathbf{C}$ is a map

$$f: T\times X \to \mathbf{C}$$

defined for some connected open subset $T\subset\mathbf{C}$ containing 0 such that

(a) for any fixed $x\in X$, $f_t(x)=f(t,x)$ is a holomorphic mapping of T to $\mathbf{C}$,

(b) for any fixed $t\in T$, f_t is an embedding, and

(c) f_0 is the identity map of X.

We think of t as a kind of complex time parameter. Note that in the definition, there is no requirement of holomorphy in the X-direction. X should be thought of with just its topological structure or its quasiconformal structure, although even continuity doesn't directly enter into the definition; the only restriction is in the t direction. We will see that continuity is a consequence of the hypotheses, by the lambda lemma of Mañé, Sad and Sullivan ([2], Theorem 2).

This definition is applicable in a number of interesting situations. For instance, the limit sets of Kleinian groups often move holomorphically as parameters are varied. Similarly, the Julia sets for iterated rational maps often move holomorphically with the parameters.

In topology, it is important to know whether an isotopy of one space within another can be extended to an ambient isotopy, that is, to an isotopy of the big space which restricts to the given isotopy of the small space. Without additional conditions, an