On Waring's problem

by

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Introduction

Landau [13] and Linnik [15], respectively, have shewn that all large numbers can be expressed as the sum of eight and of seven non-negative cubes. It is therefore a notable anomaly that no asymptotic formulae have yet been validated for the number of ways in which integers can be thus represented, the best that has been currently achieved being a formula for nine cubes. A formula for eight cubes is indeed narrowly missed by the circle method but radically new ideas would seem to be needed in order to bridge the present margin of failure.

As a contribution to the elimination of this and other lacunae in the theory of Waring's problem, we study in this memoir the effect of assuming the truth of the Riemann hypothesis for certain Hasse-Weil global L-functions defined over cubic three-folds. On this hypothesis, the precise form of which will be indicated in the text (Section 6, Chapter I), we shall indeed establish asymptotic formulae for seven and for eight cubes that are of a type previous theory would have led us to predict. But even the unconditional proof of these formulae would by no means exhaust this area of enquiry because it would still leave open the important question of the existence and number of representations of large integers as the sum of four non-negative cubes. Davenport [2], in fact, shewed that almost all numbers were representable in this manner but failed to obtain the stronger conclusion that the corresponding asymptotic formula was almost always true. We therefore partially repair this omission by deriving this formula almost always on the basis of our hypothesis, incidentally obtaining an improved estimate for the exceptional set of numbers not expressible as a sum of four non-negative cubes. As specialists in the field will recognize, this problem is in depth

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