Acta Math., 193 (2004), 241–268

Dolbeault cohomology of a loop space

by

LÁSZLÓ LEMPERT and NING ZHANG

Purdue University
West Lafayette, IN, U.S.A.

University of California
Riverside, CA, U.S.A.

0. Introduction

Loop spaces LM of compact complex manifolds M promise to have rich analytic cohomology theories, and it is expected that sheaf and Dolbeault cohomology groups of LM will shed new light on the complex geometry and analysis of M itself. This idea first occurs in [W], in the context of the infinite-dimensional Dirac operator, and then in [HBJ] that touches upon Dolbeault groups of loop spaces; but in all this, both works stay heuristic. Our goal here is rigorously to compute the Dolbeault group $H^{0,1}$ of the first interesting loop space, that of the Riemann sphere $\mathbf{P}_1$. The consideration of $H^{0,1}(L\mathbf{P}_1)$ was directly motivated by [MZ], that among other things features a curious line bundle on $L\mathbf{P}_1$. More recently, the second author classified in [Z] all holomorphic line bundles on $L\mathbf{P}_1$ that are invariant under a certain group of holomorphic automorphisms of $L\mathbf{P}_1$—a problem closely related to describing (a certain subspace of) $H^{0,1}(L\mathbf{P}_1)$. One noteworthy fact that emerges from the present research is that analytic cohomology of loop spaces, unlike topological cohomology (cf. [P, Theorem 13.14]), is rather sensitive to the regularity of loops admitted in the space. Another fact concerns local functionals, a notion from theoretical physics. Roughly, if M is a manifold, a local functional on a space of loops $x: S^1 \to M$ is a functional of form

$$f(x) = \int_{S^1} \Phi(t, x(t), \dot{x}(t), \ddot{x}(t), \dots)\, dt,$$

where Φ is a function on $S^1 \times$ an appropriate jet bundle of M. It turns out that all cohomology classes in $H^{0,1}(L\mathbf{P}_1)$ are given by local functionals. Nonlocal cohomology classes exist only perturbatively, i.e., in a neighborhood of constant loops in $L\mathbf{P}_1$; but none of them extends to the whole of $L\mathbf{P}_1$.

Both authors were partially supported by the NSF grant DMS-0203072.