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On the rational approximations to the powers of an algebraic number: Solution of two problems of Mahler and Mendès France

by

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1. Introduction

About fifty years ago Mahler [Ma] proved that *if* $\alpha>1$ *is rational but not an integer and if* $0<l<1$, *then the fractional part of* α^n *is larger than* l^n *except for a finite set of integers* n *depending on* α *and* l. His proof used a p-adic version of Roth's theorem, as in previous work by Mahler and especially by Ridout.

At the end of that paper Mahler pointed out that the conclusion does not hold if α is a suitable algebraic number, as e.g. $\frac{1}{2}(1+\sqrt{5})$; of course, a counterexample is provided by any *Pisot* number, i.e. a real algebraic integer $\alpha>1$ all of whose conjugates different from α have absolute value less than 1 (note that rational integers larger than 1 are Pisot numbers according to our definition). Mahler also added that "It would be of some interest to know which algebraic numbers have the same property as [the rationals in the theorem]".

Now, it seems that even replacing Ridout's theorem with the modern versions of Roth's theorem, valid for several valuations and approximations in any given number field, the method of Mahler does not lead to a complete solution to his question.

One of the objects of the present paper is to answer Mahler's question completely; our methods will involve a suitable version of the Schmidt subspace theorem, which may be considered as a multi-dimensional extension of the results mentioned by Roth, Mahler and Ridout. We state at once our first theorem, where as usual we denote by $\|x\|$ the distance of the complex number x from the nearest integer in $\mathbf{Z}$, i.e.

$$\|x\|:=\min\{|x-m|:m\in\mathbf{Z}\}.$$