Partitioning pairs of countable ordinals

by

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We show that the pairs of countable ordinals can be colored with uncountably many colors so that every uncountable set contains pairs of every color. This gives a definitive limitation on any form of a Ramsey Theorem for the uncountable which reduces the set of colors on some uncountable square. The first such limitation was given by Sierpiński [21] for only two colors. This was later improved by Laver (see [13]) to three colors and then by Galvin and Shelah [4] to four colors (see also Blass [1]). Our method is not based on the existence of certain uncountable linear orderings (an approach still of interest) as was the case with [21], [13], [4] and [1], but on a fine analysis of the concept of a special Aronszajn tree. This analysis will give us also a new proof of the existence of an uncountable linear ordering whose square is the union of countably many chains and many other facts about the uncountable.

All sections of this paper can be read independently from each other, but for a fuller understanding of our methods and definitions, a reading of the first three sections might be necessary. The last section contains a list of most of the recent applications of our methods as well as various other remarks concerning the previous uses of the Continuum Hypothesis in coloring pairs of countable ordinals. It should be pointed out that the main purpose of this paper is to be an exposition of our *method* of minimal walks in the realm of all *countable* ordinals because it is this case which is most often relevant to the Ramsey Problem for the uncountable. This is one of the reasons why many of the results, especially those concerning larger squares, are not stated in their full generality. Interested readers should not have any problems in formulating them in any generality they might wish to consider.

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