Regularity of gaussian processes

by

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1. Introduction

Let (Ω, Σ, P) denote a complete probability space, that will remain fixed throughout the paper. A (centered) Gaussian random variable X is a real valued measurable function on Ω such that for each real number t,

$$E \exp itX = \exp(-\sigma^2 t^2/2)$$

or that, equivalently, the law of X has a density $(2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2)$. The law of X is thus determined by $\sigma = (EX^2)^{1/2}$. If $\sigma = 1$, X is called standard normal.

A (centered) Gaussian process is a family $(X_t)_{t \in T}$ of random variables, indexed by some index set T, and such that each finite linear combination $\Sigma \alpha_t X_t$ is Gaussian. The covariance function $\Gamma(u, v) = E(X_u X_v)$ on $T \times T$ determines $E(\Sigma \alpha_t X_t)^2$, so it determines the law of the variables $(X_t)_{t \in T}$. Gaussian processes are thus a very rigid structure. One should expect, at least on philosophical grounds, that they have very nice properties. As of today, this expectation has been entirely fulfilled.

Historically, Gaussian processes, of which Brownian motion is the most important example, first occured as a model of evolution in time of a physical phenomenon. They were then naturally indexed by the real line, or by a subinterval of it. For such a process, the question of continuity arises immediately. We are dealing with an uncountable family of random variables, each of them being defined only a.e., so the very definition of continuity of the process already raises technical problems. These problems are taken care of by the use of a standard tool, the notion of "separable process". We are here hardly concerned with these technicalities, since the prime objective of this paper is to prove quantitative estimates, for which there is no loss of strength to